

Wien's Displacement Law in Rindler Space

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Abstract

In this article we have developed the formalisms for the modified form of Wien displacement laws for both the gas of electromagnetic waves and a gas of de Broglie waves in Rindler space. In the case of de Broglie waves we assume both fermion type and boson type materials. Following the classic work of Wien, we assume that the wall of the enclosure containing the photon gas or the gas of de Broglie waves, is expanding adiabatically with a uniform acceleration.

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1. INTRODUCTION

According to the principle of equivalence a frame undergoing uniform accelerated motion is equivalent to a rest frame in presence of a constant gravitational field [1–3]. Therefore considering that the surface of the enclosure which is filled up with either a photon gas or a gas of de Broglie waves is expanding adiabatically with uniform acceleration may be assumed to be equivalent to a uniformly accelerated reference frame. Then following the principle of equivalence we may replace it by a rest frame in presence of a constant gravitational field. We further assume that the gravitational field is uniform within a width ΔH inside the enclosure near the surface. Further, assuming that the structure of the enclosure is of spherical in nature, let H be the corresponding radius. It is quite obvious that a photon will gain or loose energy when it travels towards the wall or reflected back from the wall respectively [4, 5]. Hence we can say that the picture will be reversed during the reflection from the wall (see also [6] for displacement law in the case of particle production). Assuming that the wall is perfectly reflecting, then there will be no absorption of photon energy. the photon will be Doppler shifted both during its incidence and reflection at the wall. However, since the wall is expanding with uniform acceleration, unlike the case of expansion of the wall with a uniform velocity, in the present scenario the change is not so called apparent. Here the Doppler shift is associated with the real change in energy of the photons. It is therefore the gravitational redshift, not the conventional change in wavelength which occurs when there is a relative motion between the source and the observer. The wave will be blue shifted while moving towards the surface and the reverse is the case, i.e., redshift of the wave during reflection. We will show that the same is true for the matter wave associated with the incident and the reflected particles (fermions or bosons) when the enclosure is filled with either Fermi gas or Bose gas.

To get a physical insight of the incidence and the reflection processes, we consider the prescription of Feynman instead of considering Rindler space-time coordinate transformations [1, 3, 7] (see also [8–10]). These two approaches however, essentially give the same result. Moreover in our opinion the prescription of Feynman lecture is more straight forward and is easy to understand physically. Now the Rindler coordinate transformations are exactly like the Lorentz transformations in a frame undergoing uniform acceleration, otherwise in flat Minkowski space-time geometry.

In this article we have also obtained the modified form of Wien's displacement laws for de Broglie waves. The geometrical structure of the enclosure is exactly identical as we have considered in the case of photon gas.

In this article we have also studied the gravitational redshift of photons using the prescription of Friedman and Friedman et al [11–14]. The later approach is conventionally known as the extended relativistic dynamics with an upper limit of acceleration. In this study we have also tried to put forward some physical meaning to the maximum value of acceleration.

The article is organized in the following manner. In the next section, following the prescription of Feynman lecture, we have obtained the Doppler shift of photons and establish the modified form of Wien's displacement law following Saha and Srivastava [15] when the wall of the enclosure is expanding adiabatically with uniform acceleration. In section-3 we have obtained the gravitational redshift factor z following Friedman and Friedman et al [11–14]. In section-4 we have developed the Wien's displacement laws for the de Broglie waves for both fermion type and boson type particles. Finally we have given conclusion of the work.

2. WIEN'S DISPLACEMENT LAW IN RINDLER SPACE FOR PHOTON GAS

Let us consider a photon of circular frequency ω_0 , traveling towards the wall of the enclosure which is moving outward with a uniform acceleration α . The energy of the incident photon is then $\varepsilon_0 = \hbar\omega_0$. The equivalent moving mass is therefore $m_0 = \varepsilon_0/c^2 = \hbar\omega_0/c^2$. Hence the gain of energy by the photon is $m_0\alpha\Delta H$, ΔH is the distance traversed and α is assumed to be constant within ΔH in the sense of principle of equivalence. Then the energy of the photon at the instant of incidence on the wall is

$$\varepsilon_1 = \hbar\omega_0 + \frac{\hbar\omega_0}{c^2}\alpha\Delta H \quad (1)$$

Then the changed value of photon frequency is given by

$$\omega_1 = \omega_0 \left(1 + \frac{\alpha\Delta H}{c^2}\right) \quad (2)$$

This is the gravitational Doppler shifted frequency of the incident photon. The same result can also be obtained from Rindler space-time coordinate transformations. While reflected

back from the wall, the photon is traveling away from the wall. The photon will therefore loose energy. If ω_2 is the Doppler shifted frequency of the reflected photon, then the energy of this photon is given by

$$\hbar\omega_1 = \hbar\omega_2 - \frac{\hbar\omega_2}{c^2}\Delta H\alpha \quad (3)$$

Hence

$$\omega_1 = \omega_2 \left(1 - \frac{\alpha\Delta H}{c^2}\right) \quad (4)$$

Therefore during incidence it is gravitational blue shift, whereas during reflection it is gravitational redshift. Then combining eqns.(2) and (4), we have

$$\omega_2 = \omega_0 \frac{\left(1 + \frac{\alpha\Delta H}{c^2}\right)}{\left(1 - \frac{\alpha\Delta H}{c^2}\right)} \quad (5)$$

Therefore it is not the same kind of physical picture as discussed in the book by Saha and Srivastava [15]. In the later case the wall is moving with uniform velocity and the wave gets red-shifted both during incidence as well as during reflection and we know from the knowledge of optics that the change is apparent. Whereas in the case of accelerated frame of reference the energy of the photon changes both during incidence and reflection. The change of energy is proportional to the constant gravitational field. Therefore although the final result gives redshift of the photon, which is equivalent to gravitational redshift, but it is quite different from the conventional redshift of photons in optics. Since either during incidence or reflection some change in energy of the photon is occurring and it depends on the strength of constant gravitational field, the change is real in nature.

Now replacing $\omega_0 = \omega = 2\pi\nu$ and $\omega_2 = \omega + \Delta\omega = 2\pi(\nu + \Delta\nu)$, we have

$$\nu + \Delta\nu = \nu \frac{\left(1 + \frac{\alpha\Delta H}{c^2}\right)}{\left(1 - \frac{\alpha\Delta H}{c^2}\right)} \quad (6)$$

Assuming that the factor $\alpha\Delta H/c^2 \ll 1$, we have

$$\frac{\Delta\nu}{\nu} \approx \frac{2\alpha\Delta H}{c^2} \quad (7)$$

Hence we can also write

$$\frac{\Delta\lambda}{\lambda} = -\frac{2\alpha\Delta H}{c^2} \quad (8)$$

Now the single particle classical Hamiltonian is given by

$$H_0 = \left(1 + \frac{\alpha\Delta H}{c^2}\right) pc = L(\Delta H)pc = \varepsilon \quad (9)$$

where $L(\Delta H)$ is a function of ΔH . Hence following the standard result of statistical mechanics, the energy density may be written as [16]

$$\epsilon = \frac{\text{constant } T^4}{L^3(\Delta H)} \quad (10)$$

For adiabatic expansion of the photon gas

$$PV^{4/3} = \text{constant} \quad (11)$$

Again for the photon gas

$$P = \frac{1}{3}\epsilon \quad (12)$$

Hence

$$\epsilon V^{4/3} = \text{constant} \quad (13)$$

The above equation can also be expressed as

$$\frac{T^4 H^4}{\left(1 + \frac{\alpha \Delta H}{c^2}\right)^3} = \text{constant} \quad (14)$$

where we have used $V = 4\pi H^3/3$, the volume of the enclosure. Hence assuming that the factor $\alpha \Delta H/c^2 \ll 1$, we have

$$HT = \text{constant} \quad (15)$$

Now for the oblique incidence, we can write

$$\frac{\Delta \lambda}{\lambda} = -\frac{2\alpha \Delta H}{c^2} \cos \theta \quad (16)$$

where θ is the angle of incidence. This is the change in wavelength per collision. Then following Saha and Srivastava [15], we have the change of wavelength per unit time

$$\frac{\Delta \lambda}{\lambda} = -\frac{\alpha}{c} \frac{\Delta H}{H} \quad (17)$$

In the limiting case, we can write

$$\frac{d\lambda}{\lambda} = -\frac{\alpha}{c} \frac{dH}{H} \quad (18)$$

Integrating, we have

$$\lambda H^{\omega_c} = \text{constant} \quad (19)$$

where $\omega_c = \alpha/c$, the frequency of cosmic phonons as described in [17]. Then combining eqns.(15) and (19), we have

$$\lambda T^{-\omega_c} = \text{constant} \quad \text{or} \quad \lambda \propto T^{\omega_c} \quad (20)$$

This is the modified version of Wien's displacement law for photon gas in the Rindler space. It is therefore obvious that λ will be quite large if α is large enough. This is consistent with the conventional result on gravitational redshift. Further, the wavelength saturates to a constant value for $\alpha \rightarrow 0$.

3. GRAVITATIONAL REDSHIFT IN EXTENDED RELATIVISTIC DYNAMICS

Next we consider the prescription of Friedman and Friedman et al [11–14] on the extended relativistic dynamics and maximal acceleration. Now the clock hypothesis of Einstein states that the timing of an accelerated clock is identical with that of a clock at rest in some un-accelerated frame of reference. It is well known that if the clock hypothesis is correct, the transformations are Galilean type. Whereas, if the clock hypothesis is false, then the transformations are Lorentz type. In the later case the uniform acceleration of the frame plays the role of uniform velocity of Lorentz type transformations. The maximal value of acceleration plays the role of velocity of light. In the special theory of relativity the velocity of light is treated as the upper limit of velocity.

Then following Friedman, when the clock hypothesis is false, we have the proper velocity-time transformations

$$\begin{aligned} t &= \gamma \left(t' + \frac{\alpha u'_x}{\alpha_m^2} \right) \\ u_x &= \gamma (\alpha t' + u'_x) \\ u_y &= u'_y \\ u_z &= u'_z \end{aligned} \tag{21}$$

where $\gamma = (1 - \alpha^2/\alpha_m^2)^{1/2}$, the time dilation factor. The motion is assumed to be along x -direction and the uniform acceleration α is also along the same direction. Here α_m is the upper limit of acceleration, just like the velocity of light is the maximum possible value for velocity in the case of Lorentz transformation in special theory of relativity.

To get an estimate for Doppler shift of an electromagnetic wave, we follow Friedman and assume that the wave-vector k in the proper velocity-time representation is also along x -direction. Then from [11], the electromagnetic radiation may be represented by the function

$$(\omega t - k u) = f \left[\omega \gamma \left(t' + \frac{\alpha u'}{\alpha_m^2} \right) - k \gamma (\alpha t' + u') \right]$$

$$\begin{aligned}
&= f \left[\gamma (\omega - k\alpha) t' - \gamma \left(k - \frac{\alpha}{\alpha_m^2} \right) u' \right] \\
&= f(\omega' t' - k' u') \quad (\text{say})
\end{aligned} \tag{22}$$

Hence

$$\begin{aligned}
\omega' &= \gamma (\omega - k\alpha) \\
&= \omega \frac{\left(1 - \frac{\alpha}{\alpha_m} \right)}{\left(1 - \frac{\alpha^2}{\alpha_m^2} \right)^{1/2}}
\end{aligned} \tag{23}$$

For $\alpha/\alpha_m \ll 1$ and writing $\omega = 2\pi\nu$, with ν the actual frequency, we have

$$\nu' = \nu \left(1 - \frac{\alpha}{\alpha_m} \right) \tag{24}$$

Following the discussion of section-2, here also we consider the outward expansion of the wall of the enclosure, containing photon gas, with a uniform acceleration α . In this prescription, when a photon of frequency ν incident on the surface of the enclosure, it will be red-shifted to ν' as given in eqn.(24). Further, if the surface is perfect reflector, the wave will again be red-shifted without any loss of energy by absorption and the final frequency will be

$$\nu'' = \nu \frac{\left(1 - \frac{\alpha}{\alpha_m} \right)}{\left(1 + \frac{\alpha}{\alpha_m} \right)} \tag{25}$$

Now writing $\nu'' = \nu + d\nu$, where $d\nu$ is the effective change in frequency due to Doppler shift, we have

$$\frac{d\nu}{\nu} = -\frac{2\alpha}{\alpha_m} \tag{26}$$

Hence

$$\frac{d\lambda}{\lambda} = \frac{2\alpha}{\alpha_m} \tag{27}$$

Here we have considered only normal incidence. Now eqn.(27) may be approximated with z , the gravitational redshift factor. Here the quantity z depends only on the uniform acceleration α . Further the redshift arises because of the accelerated motion of the reflecting surface. In accordance with the principle of equivalence, we may replace the uniformly accelerated frame by a stationary frame in presence of a constant gravitational field α . Therefore one may assume that α is the constant gravitational field produced by a self gravitating object, e.g., a neutron star (not a black hole, because nothing is known about the gravitational field of the black hole). Then α_m is the maximum possible surface value of gravitational field for

a neutron star. For $\alpha = \alpha_m$, the gravitational redshift factor z becomes 2. Since α is always less than α_m , therefore z is always less than 2, the bound for gravitational redshift factor at the neutron star surface for $M/R < 4/9$ [18] in geometrical unit, where M and R are respectively the mass and radius of a neutron star. Therefore according to our analysis, the quantity α_m has physical significance. It is related to the maximum gravitational redshift for the compact stellar object concerned.

4. WIEN'S DISPLACEMENT LAW FOR DE BROGLIE WAVES IN RINDLER SPACE

To develop a formalism for Wien's displacement law of de Broglie waves in Rindler space we replace $\hbar\omega$ by the single particle energy ε of the fermion or boson and further we write in the non-relativistic approximation the single particle energy $\varepsilon = p^2/2m$, where m is the particle mass. In this model calculation we assume that the enclosure whose wall is expanding adiabatically with constant acceleration is filled up with either a Fermi gas or a Bose gas. At first we develop the formalism in a very general manner, applicable for both the fermions and bosons. At the end of this section only we shall differentiate between fermions from bosons. Further, the collisions of the particles are assumed to be elastic in nature.

Now following eqns.(1) and (3), we have

$$p_1 = p_0 \left(1 + \frac{\alpha\Delta H}{c^2}\right)^{1/2} \approx p_0 \left(1 + \frac{\alpha\Delta H}{2c^2}\right) \quad (28)$$

and

$$p_1 \approx p_2 \left(1 - \frac{\alpha\Delta H}{2c^2}\right) \quad (29)$$

Combining these two equations, we have

$$P_2 = p_0 \frac{1 + \frac{\alpha\Delta H}{2c^2}}{1 - \frac{\alpha\Delta H}{2c^2}} \quad (30)$$

Now writing $\lambda = h/p$, the de Broglie wave length of the particle, we have

$$\lambda_2 = \lambda_0 \frac{1 - \frac{\alpha\Delta H}{2c^2}}{1 + \frac{\alpha\Delta H}{2c^2}} \quad (31)$$

This is the relation between the initial (before incidence) and final (after reflection) de Broglie wavelengths of the particle. Now as before, assuming an infinitesimal change in de

Broglie wavelength during reflection from the wall, we can write $\lambda_2 = \lambda + \delta\lambda$, where we have put λ for λ_0 , the initial de Broglie wavelength and $\delta\lambda$ is the infinitesimal change in de Broglie wavelength. Then following the same procedure as has been used for photon gas, we have in the limiting case

$$\frac{d\lambda}{\lambda} = -\omega_c \frac{dH}{H} \quad (32)$$

On integrating, we have

$$\lambda H^{\omega_c} = \text{constant} \quad (33)$$

Now for a non-relativistic gas

$$P = \frac{2}{3}\epsilon \quad (34)$$

where ϵ is the energy density and P is the kinetic pressure of the gas. Now for the adiabatic expansion of non-relativistic gas, we have

$$PV^{5/3} = \text{constant} \quad (35)$$

Combining these two equations, we can write

$$\epsilon V^{5/3} = \text{constant} \quad (36)$$

Now for a non-relativistic gas obeying Boltzman statistics, we can write

$$\epsilon = CT^{5/2} \exp\left(\frac{\mu}{kT}\right) \quad (37)$$

where C is a constant and μ is the chemical potential of the gas. Combining eqns.(36) and (37) and using $V = 44\pi H^3/3$ as the volume of the enclosure, with H its radius, we have

$$TH^2 \exp\left(\frac{2\mu}{5kT}\right) = \text{constant} \quad (38)$$

Again using eqn.(33), we have for a Fermi gas with non-zero chemical potential

$$\lambda = \text{constant} T^{0.5\omega_c} \exp\left(\frac{\mu\omega_c}{5kT}\right) \quad (39)$$

This is the modified form of Wien's displacement law for fermionic de Broglie wavelength in Rindler space. Obviously the wavelength increases in a much faster rate with the strength of gravitational field α in comparison with photon gas for fixed T . Further, the wavelength exponentially diverges for large α . Similar to the photon gas the de Broglie wavelength saturates to a constant value for very low α ($\alpha \rightarrow 0$). On the other hand if one considers

an anti-fermion with chemical potential $-\mu$, then the de Broglie wavelength after reflection becomes

$$\lambda = \text{constant} T^{0.5\omega_c} \exp\left(\frac{-\mu\omega_c}{5kT}\right) \quad (40)$$

Therefore as the gravitational field becomes strong enough the de Broglie wavelength of the anti-fermion decreases and in the extreme case for $\alpha \rightarrow \infty$, $\lambda \rightarrow 0$. Therefore in extremely strong gravitational field anti-fermions behave like classical objects. Since anti-particles do not exist in classical mechanics, even in the non-relativistic quantum mechanics they do not exist. In Schrödinger equation particle and the corresponding anti-particle are treated in equal footing. Therefore we can conclude that Strong gravitational field will never allow the emission of anti-particles. When the pairs are produced at the surface of a strongly gravitating object, then we expect only the particle will be emitted, whereas the anti-particle counter part will go inside the object. Which may be the another version of Penrose mechanism, but the reason is quite different.

Next we consider a Bose gas with chemical potential $\mu = 0$ (say a mixture of π^+ and π^- .) Then it is obvious that the modified form of Wien's displacement law is given by

$$\lambda = \text{constant} T^{0.5\omega_c} \quad (41)$$

The variation is therefore little slower compared to photon gas.

5. CONCLUSION

In this article we have developed a formalism to study Wien's displacement law for photon gas and a gas of de Broglie waves in a frame undergoing a uniform accelerated motion.

It is well known that the conventional form of Wien's displacement law for photon gas is $\lambda \propto T^{-1}$, whereas in the present scenario it is $\lambda \propto T^{\omega_c}$, with $\omega_c = \alpha/c \geq 0$. Hence one can infer that for $\alpha \rightarrow \infty$, $\lambda \rightarrow \infty$. This is consistent with the results of gravitational redshift in presence of strong gravitational field produced by some compact massive stellar objects. In this case the redshift is because of uniform acceleration of the frame or because of constant gravitational field in the sense of principle of equivalence. Therefore not the temperature, but the strength of gravitational field plays the major role in producing Doppler shift of the wave if the field is extremely strong. It is also obvious that for low gravitational field (acceleration), i.e., for $\alpha \rightarrow 0$, the wavelength λ saturates to some constant value. This is

true for the formalism either based on the prescription of Feynman lecture or using Rindler coordinates.

In the second part we have studied the gravitational redshift for electromagnetic waves in the context of extended relativistic dynamics with an upper limit of acceleration α_m of the frame. In this scenario we have noticed that for $\alpha = \alpha_m$, the gravitational redshift factor $z = 2$. Since α is always $< \alpha_m$, we have $z < 2$, the bound of gravitational redshift factor. This can also be obtained from general theory of relativity with $M/R < 4/9$ [19].

In the third part we have studied the gravitational redshift of de Broglie waves in Rindler space. In the case of fermionic de Broglie waves the effect is more prominent because of the exponential term. The wavelength increases much faster with the gravitational field compared to photon gas. For extremely high gravitational field the wavelength diverges exponentially. Whereas for a Bose gas with zero chemical potential the increase of wavelength with gravitational field is little bit slower compared to photon gas. For extremely high field it again diverges, but not exponentially.

The physical reason for the gravitational redshift of photons is the curvature of the space produced by gravity, whereas in the case of de Broglie waves it is because of the gain or loss of energy of the particle, when it goes in favor or against gravity respectively.

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